A Fixed Point Theorem for a Selfmap of a Compact S-Metric Space

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ABSTRACT

The purpose of this paper is to prove a fixed point theorem for selfmap of a compact S-metric space, and we show that a fixed point theorem of metric space proved by Brain Fisher ([5], Theorem 2) follows as a particular case of our theorem

MSC: *47H10, 54H25*

Key words: *S-metric space; Fixed point theorem*

INTRODUCTION AND PRELIMINARIES

Metric space is one of the most useful and important space in mathematics. Its wide area provides a powerful tool to the study of variational inequalities, optimization and approximation theory, computer sciences and so many. Recently the study of fixed point theory in metric space is very interesting and attract many researchers to investigated different results on it.

On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gahler [6] gave the concepts of 2- metric space further in 1992 Dhage [2] modified the concept of 2-metric space and introduced the concepts of D-metric space also proved fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [7], [8] and [9]. Sedghi et al. [10] modified the concepts of D- metric space and introduced the concepts of D*- metric space also proved a common fixed

point theorems in D*- metric space.

Recently, Sedghi et al [11] introduced the concept of S- metric space which is different from other space and proved fixed point theorems in S-metric space. They also gives some examples of S- metric spaces which shows that S- metric space is different from other spaces. In fact they gives following concepts of S- metric space.

Definition 1.1([11]): Let X be a non-empty set. An S-metric space on X is a function $S: X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$

(i) $S(x, y, z) \geq 0$
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$.
(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X, S) is called an **S–metric space**.
Immediate examples of such S-metric spaces are:

Example1.2: Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - y| + |y - z| + |z - x|$ for each $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Example 1.3: Let $X = \mathbb{R}^2$, d be the ordinary metric on X.
Put $S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ is an S- metric on X. If we connect the points x, y, z by a line, we have a triangle and if we choose a point a mediating this triangle then the inequality $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ holds. In fact

$$S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$
$$\leq d(x, a) + d(a, y) + d(y, a) + d(a, z) + d(z, a) + d(a, x)$$
$$= S(x, x, a) + S(y, y, a) + S(z, z, a)$$

Example1. 4: Let $X = \mathbb{R}^n$ and $\|\,.\,\|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Remark1. 5: it is easy to see that every D*-metric is S-metric, but in general the converse is not true, see the following example.

Example1. 6: Let $X = \mathbb{R}^n$ and $\|\,.\,\|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X, but it is not D*-metric because it is not symmetric.

Lemma1. 7: In an S–metric space, we have $S(x, x, y) = S(y, y, x)$.

Proof: By the third condition of S-metric, we get
$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x) = S(y, y, x)$…… (1)
and similarly
$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y) = S(x, x, y)$……(2)
Hence, by (1) and (2), we obtain $S(x, x, y) = S(y, y, x)$.

Definition1.8: Let (X, S) be an S-metric space. For $x \in X$ and $r > 0$, we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with a center x and a radius r as follows

$$B_S(x, r) = \{y \in X;\ S(x, y, y) < r\}$$
$$B_S[x, r] = \{y \in X;\ S(x, y, y) \leq r\}$$

For example, Let $X = \mathbb{R}$. Denote $S(x, y, z) = |\,y + z - 2x\,| + |\,y - z\,|$ for all $x, y, z \in \mathbb{R}$. Therefore $B_S(1, 2) = \{y \in \mathbb{R}\ ;\ S(y, y, 1) < 2\}$

$$= \{y \in \mathbb{R}\ ;\ |\,y - 1| < 1\} = (0, 2).$$

Definition 1.9: Let (X, S) be an S–metric space and $A \subset X$.

i. If for every $x \in A$, there is a $r > 0$ such that $B_S(x, r) \subset A$, then the subset A called an **open subset** of X
ii. If there is a $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$ then A is said to be **S–bounded.**
iii. A sequence $\{x_n\}$ in X **converge to x** if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote this by $\lim_{n \to \infty} x_n = x$
iv. A sequence $\{x_n\}$ in X is called a **Cauchy sequence** if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$
v. The S–metric space (X, S) is said to be **complete** if every Cauchy sequence is convergent sequence.
vi. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).
vii. If (X, τ) is a compact topological space we shall call (X, S) is a **compact** S–metric space.

Lemma1. 10 ([11]): Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the open ball $B_S(x, r)$ is an open subset of X.

Lemma1. 11([11]**):** Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma1. 12([11]**):** Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.

Lemma1. 13([11]**):** Let (X, S) be an S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma1. 14: Let (X, d) be a metric space. Then we have

- **i.** $S_d(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for all $x, y, z \in X$ is an S-metric on X
- **ii.** $x_n \to x$ in (X, d) if and only if $X_n \to x$ in (X, S_d)
- **iii.** $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, S_d)
- **iv.** (X, d) is complete if and only if (X, S_d) is complete

Proof:

- i. See [Example (3), Page 260]
- ii. $x_n \to x$ in (X, d) if and only if $d(x_n, x) \to 0$, if and only if $S_d(x_n, x_n, x) = 3d(x_n, x) \to 0$ that is, $x_n \to x$ in (X, S_d)
- iii. $\{x_n\}$is a Cauchy in (X, d) if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$, if and only if $S_d(x_n, x_n, x_m) = 3d(x_n, x_m) \to 0$ $n, m \to \infty$, that is, $\{x_n\}$ is Cauchy in (X, S_d)
- iv. It is a direct consequence of (2) and (3)

Notation: For any selfmap T of X, we denote T(x) by Tx.
If P and Q are selfmaps of a set X, then any $z \in X$ such that Pz = Qz = z is called a **common fixed point** of P and Q. Two selfmaps P and Q of X are said to be **commutative** if PQ = QP where PQ is their composition PoQ defined by (PoQ) x = PQx for all $x \in X$.

THE MAIN RESULT

Theorem 2.1:- If T is a selfmap of a S-metric space (X, S) into itself such that

(i) there is a $z \in X$, with $S(z, Tz, Tz) \leq S(x, Tx, Tx)$ for all $x \in X$,

and

(ii) $S(Tx, Ty, Ty) < \frac{1}{2}[S(x, Ty, Ty) + S(y, Tx, Tx)]$ for all $x, y \in X$, with $x \neq y$.

Then T has a unique fixed point.

Proof: Suppose $T: X \to X$ is such that $f(z) = S(z, Tz, Tz) \leq S(x, Tx, Tx) = f(x)$ for all $x \in X$.

That is, $f(z) \leq f(x)$ for all $x \in X$. Now we claim that Tz = z.

If $Tz \neq z$, then by (ii) we have

$$\begin{aligned} S(Tz, T^2z, T^2z) &< \tfrac{1}{2}[S(z, T^2z, T^2z) + S(Tz, Tz, Tz)] \\ &= \tfrac{1}{2} S(z, T^2z, T^2z) \\ &< \tfrac{1}{2}[S(z, Tz, Tz) + S(Tz, T^2z, T^2z)], \end{aligned}$$

which implies $S(Tz, T^2z, T^2z) < S(z, Tz, Tz)$. That is, $f(Tz) < f(z)$, a contradiction to the definition of f(z). Hence Tz = z, giving z is a fixed point of T.
Now suppose that T has another fixed point, say z' with $z \neq z'$, then by (ii) we have

$$S(z, z', z') = S(Tz, Tz', Tz') < \tfrac{1}{2}[S(z, Tz', Tz') + S(z', Tz, Tz)],$$

which implies $S(z, z', z') < S(z, z', z')$, a contradiction. Therefore we have z = z'. That is z is the unique fixed point of T.

As a consequence we have a fixed point theorem for a selfmap of a compact S-metric space.

Corollary 2.2:- If T is a selfmap of a compact S-metric space (X, S) such that
$S(Tx, Ty, Ty) < \frac{1}{2}$ [S(x, Ty, Ty) + S(y, Tx, Tx)] for all x, y $\in$ X, with $x \neq y$. Then T has a unique fixed point.

Proof: Writing f(x) = S(x, Tx, Tx) for x $\in$ X, it can be seen that f is a continuous real function on X. Since X is compact, f attains minimum at some z $\in$ X. That is $f(z) \leq f(x)$ for all x $\in$ X. That is, z $\in$ X is such that S(z, Tz, Tz) $\leq$ S(x, Tx, Tx) for all x $\in$ X, which is the condition (i) of Theorem 2.1. Also by the hypothesis, condition (ii) of Theorem 2.1 holds. Hence the corollary follows from Theorem 2.1.
Now we deduce the following theorem

Corollary2.3 ([5] Theorem 2):- If T is a selfmap of a metric space (X, d) into itself such that
(i) T is continuous
(ii) X is compact,
and
(iii) $d(Tx, Ty) < \frac{1}{2}[d(x, Ty) + d(y, Tx)]$ for all x,y $\in$ X, with $x \neq y$,
then T has a unique fixed point.

Proof: Given (X, d) is a metric space. Then
S(x, y, z) =max {d(x, y), d(y, z), d(z, x)}, then(X, S) is a S-metric space on X and
S(x, y, x) = d(x, y). Therefore, condition (iii) gives

$$S(Tx, Ty, Tx) < \frac{1}{2}[S(x, Ty, x) + S(y, Tx, y)]$$
$$= \frac{1}{2}[S(x, Ty, Ty) + S(y, Tx, Tx)]$$

which is the same as condition of Corollary 2.2. Also since (X, d) is complete, we have that (X, S) is complete, hence the corollary follows.

REFERENCES

Ahmed, B., Ashraf, M., & Rhoades, B. E. (2001). Fixed point theorems for expansive mappings in D-metric spaces. *Indian Journal of Pure and Applied Mathematics, 30*(10), 1513–1518.

Dhage, B. C. (1992). Generalised metric spaces and mappings with fixed point. *Bulletin of the Calcutta Mathematical Society, 84*(4), 329–336.

Dhage, B. C. (1999). A common fixed-point principle in D-metric spaces. *Bulletin of the Calcutta Mathematical Society, 91*(6), 475–480.

Dhage, B. C., Pathan, A. M., & Rhoades, B. E. (2000). A general existence principle for fixed point theorems in D-metric spaces. *International Journal of Mathematics and Mathematical Sciences, 23*(7), 441–448. https://doi.org/10.1155/S0161171200003112

Fisher, B. (1978). A fixed-point theorem for compact metric spaces. *Publicationes Mathematicae Debrecen, 25*, 193–194.

Gähler, S. (1963). 2-metrische Räume und ihre topologische Struktur. *Mathematische Nachrichten, 26*(1-4), 115–148. https://doi.org/10.1002/mana.19630260109

Naidu, S. V. R., Rao, K. P. R., & Srinivasa Rao, N. (2004). On the topology of D-metric spaces and generalization of D-metric spaces from metric spaces. *International Journal of Mathematics and Mathematical Sciences, 2004*(51), 2719–2740. https://doi.org/10.1155/S0161171204401631

Naidu, S. V. R., Rao, K. P. R., & Srinivasa Rao, N. (2005). On the concepts of balls in a D-metric space. *International Journal of Mathematics and Mathematical Sciences, 2005*(1), 133–141. https://doi.org/10.1155/IJMMS.2005.133

Naidu, S. V. R., Rao, K. P. R., & Srinivasa Rao, N. (2005). On convergent sequences and fixed-point theorems in D-metric spaces. *International Journal of Mathematics and Mathematical Sciences, 2005*(12), 1969–1988. https://doi.org/10.1155/IJMMS.2005.1969

Sedghi, S., Shobe, N., & Aliouche, A. (2012). A generalization of fixed point theorem in S-metric spaces. *Matematicki Vesnik, 64*(3), 258–266.

Sedghi, S., Shobe, N., & Zhou, H. (2007). A common fixed-point theorem in D*-metric spaces. *Fixed Point Theory and Applications, 2007*, Article 27906. https://doi.org/10.1155/2007/27906